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LETTER TO THE EDITOR

Simple excitations in the nested Bethe-ansatz model

Junji Suzuki

Instittue of Physics, College of Arts and Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, Japan

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Abstract. The scaling dimension of the operators in the nested Bethe ansatz model are studied. They explicitly show the structure of the underlying Lie algebras; they can be written in terms of the Cartan matrix. We infer that they are the natural extension of those of SU(2), or equivalently of the Coulomb gas.

Recently conformal invariance has become one of the most dominant concepts in the study of critical phenomena. Among the remarkable outcomes of conformal invariance theory, the formulae derived by Cardy [1], Blöte *et al* [2] and Affleck [3] turn out to be extremely fruitful in statistical mechanics. They showed that the central charge and the operator dimensions can be derived by evaluating finite-size corrections. On the other hand, de Vega and Woynarovich [4] invented an efficient method of evaluating the finite correction to the free energy in the case that the model can be exactly solved by the Bethe ansatz. Their method was refined by Woynarovich and Eckle [5] so as to enable one to evaluate higher correction terms. Several models were analysed by this method, e.g. the *XXZ* chain, 6-vertex [6], the higher spin chain [7], 8-vertex [8], Potts, Ashkin–Teller [9], O(n) [10], the Hubbard model [11], etc.

Based on these works, we study in this letter the scaling dimensions of some simple excitations in the nested Bethe-ansatz (NBA) model [12] having the rational S matrix. It is known that a large number of theoretical models have rational S matrices and, further, as shown by Ogievetsky and Wiegmann [13], the existence of the rational S matrix necessarily leads to the NBA, so that our approach applies to a great many important models such as the SU(N) magnetic chain [14], O($2k$) Gross–Neveu model [15], etc. The resulting expressions for the dimensions of scaling and spin operators show the structure of the underlying Lie algebras explicitly, and are seen to be natural extensions to those of SU(2) or, equivalently, 2D Coulomb gas. Thus the outcome of the present work may hopefully shed some light on the further study of the operator content of the NBA model.

To work with the NBA model we generalise the methods of [4] and [5] to multicomponent systems. The NBA model having the rational S matrix is characterised by

$$N\phi(\lambda_{j_k}^{(k)},\omega_k)-2\pi I_{j_k}=\sum_{l=1}^{r}\sum_{j_l=1}^{n_l}\phi(\lambda_{j_k}^{(k)}-\lambda_{j_l}^{(l)},\,\omega_{lk}). \tag{1}$$

In the above equation, N is the linear size of the system, n_l is the number of 'particles' having colour l, r is the rank of the underlying Lie algebra, $\phi(z,\alpha)=\mathrm{i}\log[(z+\alpha)/(z-\alpha)]$ and $\omega_k=(\omega,\alpha_k)$, $\omega_{lk}=(\alpha_l,\alpha_k)$ where ω is the highest weight and α_j is the simple root.

The free energy for the finite system is given by

$$f(\theta)=-\frac{\mathrm{i}}{N}\sum_{k=1}^{r}\sum_{j_k=1}^{n_k}\phi(\lambda_{j_k}^{(k)}+\mathrm{i}\theta,\omega_k) \tag{2}$$

where θ corresponds to the spectral parameter.

In the thermodynamic limit ($N\to\infty$), the root densities $\sigma_\infty^{(k)}(\lambda)$ (here 'root' means the solution to the NBA equation; be careful not to confuse with the 'root' used in the Lie algebra) are given by the solutions to the following integral equations:

$$\frac{1}{2\pi}\phi'(\lambda,\omega_l)=\sigma_\infty^{(l)}(\lambda)+\sum_k\int_{-\infty}^{\infty}\mathrm{d}\mu\,K_{lk}(\lambda-\mu)\sigma_\infty^{(k)}(\mu) \tag{3}$$

where $K_{lk}(\lambda)=\phi'(\lambda,\omega_{lk})/2\pi$.

The solution can be written in terms of the Green matrix $R_{lk}(\lambda)$ as

$$\sigma_\infty^{(l)}(\lambda)=\sum_k\int\mathrm{d}\mu\,R_{lk}(\lambda-\mu)\phi'(\mu,\omega_k)/2\pi. \tag{4}$$

The Fourier-transformed form of R^{-1} plays an important part in the following; its explicit form is

$$(\tilde{R}^{-1})_{lk}(x)=\delta_{lk}+\mathrm{sgn}[(\alpha_l,\alpha_k)]\exp(-|(\alpha_l,\alpha_k)x|) \tag{5}$$

with $\mathrm{sgn}(0)=0$. Since α_j is the simple root, $\tilde{R}^{-1}$ at $x=0$ is nothing but the Cartan matrix $\mathscr{C}$ for the underlying Lie algebra:

$$\tilde{R}^{-1}(0)=\mathscr{C}. \tag{6}$$

The free energy can be written in terms of $\sigma_\infty^{(l)}(\lambda)$ as

$$f_\infty(\theta)=-\mathrm{i}\sum_k\int\mathrm{d}\lambda\,\phi(\lambda+\mathrm{i}\theta,\omega_k)\sigma_\infty^{(k)}(\lambda). \tag{7}$$

As usual, we define the root density functions for the large but finite system as

$$\sigma_N^{(l)}(\lambda)=\mathrm{d}z_N^l(\lambda)/\mathrm{d}\lambda\qquad l=1,\ldots,r \tag{8}$$

where z_N^l is defined as a function of λ through

$$z_N^l(\lambda)=\frac{1}{2N}\left(\phi(\lambda,\omega_l)-\frac{1}{N}\sum_{k,j_k}\phi(\lambda-\lambda_{j_k}^{(k)},\omega_{lk})\right). \tag{9}$$

For n_e^l, the number of particles of colour l in the excited state we introduce the maximum roots Λ_l by solving

$$z_N^l(\Lambda_l)=n_e^l-1/2N\qquad l=1,\ldots,r. \tag{10}$$

Integration of (8) with the help of (4), (6), (9) and (10) yields

$$\int_{\Lambda_l}^{\infty}\sigma_N^{(l)}(\mu)\,\mathrm{d}\mu=\frac{1}{2N}\sum_k(\mathscr{C}_{lk}\boldsymbol{n}_k)+1/2N \tag{11}$$

where $\boldsymbol{n}_l=n_g^l-n_e^l$ with n_g^l being the number of particles of colour l in the ground state. Since we can, in principle, assign any integer to n_e^l, we can choose the Λ_l arbitrarily, but in this letter we treat the simplest case, $\Lambda_1=\Lambda_2=\ldots=\Lambda_r=\Lambda$ to avoid mathematical complexities. As shown later, all excitations are massless in this case.

The following equations can be easily derived from equations (1)-(8):

$$\sigma_N^{(l)}(\lambda)-\sigma_\infty^{(l)}(\lambda)=-\sum_{m,p}\int_{-\infty}^{\infty} R_{lp}(\lambda-\mu)K_{pm}(\mu-\xi)S_N^m(\xi)\,\mathrm{d}\xi\,\mathrm{d}\mu \tag{12}$$

$$f_N(\theta)-f_\infty(\theta)=\mathrm{i}\sum_{l,m}\int_{-\infty}^{\infty}\phi(\lambda+\mathrm{i}\theta,\omega_l)R_{lm}(\lambda-\mu)S_N^m(\mu)\,\mathrm{d}\lambda\,\mathrm{d}\mu \tag{13}$$

where S_N^m is defined by

$$S_N^m(\lambda)=\frac{1}{N}\sum_{j_m}\delta(\lambda-\lambda_{j_m}^{(m)})-\sigma_N^{(m)}(\lambda). \tag{14}$$

Let us define $X_l^\mp$ and $F_l^\mp$ respectively through

$$X_l^\pm(\lambda)=\Theta(\pm\lambda)\sigma_N^{(l)}(\lambda+\Lambda) \tag{15}$$

$$F_l^\pm(\lambda)=\Theta(\pm\lambda)\sigma_\infty^{(l)}(\lambda+\Lambda) \tag{16}$$

where $\Theta(x)$ denotes the conventional step function. Then (12) can be rewritten into a Wiener–Hopf type integral equation. Indeed in the case $\lambda\sim\Lambda$, it can be written in the Fourier-transformed form as

$$\tilde{R}(x)\tilde{\boldsymbol{X}}^+(x)+\tilde{\boldsymbol{X}}^-(x)-(\tilde{\boldsymbol{F}}^+(x)+\tilde{\boldsymbol{F}}^-(x))=\tilde{K}(x)\tilde{R}(x)\boldsymbol{u}(x) \tag{17}$$

where the jth component of $\boldsymbol{u}(x)$ is given by

$$(\boldsymbol{u}(x))_j=\frac{1}{2N}+\frac{\mathrm{i}x}{12N\sigma_N^{(j)}(\Lambda)}. \tag{18}$$

In order to solve (17), we first verify the following properties of the $\tilde{R}^{-1}(x)$:

$$[\tilde{R}^{-1}(x),\tilde{R}^{-1}(y)]=0 \tag{19a}$$

$$\tilde{R}^{-1}(x=\infty)=E. \tag{19b}$$

They ensure that there exists an x-independent orthogonal matrix $\mathscr{D}$ which diagonalises $\tilde{R}^{-1}(x)$, i.e.

$$\tilde{R}^{-1}(x)=\mathscr{D}\mathscr{T}(x)^t\mathscr{D}. \tag{20}$$

$\mathscr{T}(x)$ is a diagonal matrix, the components of which are eigenvalues $\tau_j(x)$ ($j=1,\ldots,r$) of $\tilde{R}^{-1}(x)$. Then, following the standard method [16], we represent $\tilde{R}^{-1}(x)$ as the product of the matrices $\Xi_+(x)$ and $\Xi_-(x)$ which are analytic in the upper and the lower half planes respectively as

$$\tilde{R}^{-1}(x)=\Xi_+(x)\Xi_-(x) \tag{21}$$

under the restrictions

$$\Xi_-(x)=(\Xi_+(x))^\dagger \tag{22a}$$

and

$$\lim_{|x|\to\infty}\Xi_+(x)=E. \tag{22b}$$

In order to obtain the explicit form for Ξ_+, we decompose τ_j as

$$\tau_j(x)=\tau_j^+(x)\tau_j^-(x) \tag{23}$$

where $\tau_j^+(x)$ ($\tau_j^-(x)$) is an analytic function in the upper (lower) half plane, and their forms are given by

$$\tau_j^\pm(x)=\left(\mp\mathrm{i}\frac{x}{4\pi e}\right)^{\mp\mathrm{i}x/2\pi}\frac{(\theta_j\mp\mathrm{i}x/2)}{\Gamma(1\mp\mathrm{i}x/4\pi-\theta_j/2\pi)\Gamma(1\mp\mathrm{i}x/4\pi+\theta_j/2\pi)}. \tag{24}$$

In the above θ_j denotes the properly normalised Coxeter exponents for the underlying Lie algebras (see table 1). Asymptotically they behave as

$$\tau_j^\pm(x)\sim 1\pm g_1^j/x+g_2^j/x^2 \qquad \text{as } |x|\to\infty$$

with

$$\begin{aligned} g_1^j&=2\mathrm{i}\pi(\theta_j/\pi-\tfrac{1}{3}-\theta_j^2/2\pi^2)\\ g_2^j&=(g_1^j)^2/2. \end{aligned} \tag{25}$$

From equations (20)-(22) we obtain the explicit form for $\Xi_\pm$ as

$$\Xi_\pm=\mathscr{D}\mathscr{M}_\pm(x)'\mathscr{D} \tag{26}$$

where $(\mathscr{M}_\pm(x))_{i,j}=\delta_{i,j}\tau_j^\pm(x)$. One can easily see that both ($22a$) and ($22b$) are satisfied. With the matrix Ξ_+ obtained above, the solution to (17), $\tilde{\boldsymbol{X}}_+$, is written in the form

$$\tilde{\boldsymbol{X}}_+(x)=\boldsymbol{u}(x)+\Xi_+(x)(\boldsymbol{Q}_+(x)+\boldsymbol{P}(x)). \tag{27}$$

In the above, $\boldsymbol{Q}_+(x)$ denotes

$$\boldsymbol{Q}_+(x)=\frac{\kappa}{2\pi(\kappa-\mathrm{i}x)}\,{}'\Xi_+(\mathrm{i}\kappa)\boldsymbol{w} \tag{28}$$

Table 1. List of quantities κ, m_j, θ_j for various Lie algebras. See the text for definition.

Lie algebra	κ	m_j	(θ_j) $(j=1,\ldots,n)$
A_n	$\frac{2x}{n+1}$	$\sin\left(\frac{j\pi}{n+1}\right)$ $j=1,\ldots,n$	$\frac{j\pi}{n+1}$ $(j=1,\ldots,n)$
D_n	$\frac{\pi}{n-1}$	$\begin{cases}2\sin\left(\frac{j\pi}{2(n-1)}\right) & j=1,\ldots,n-2\\ m_\pm=1\end{cases}$	$\begin{cases}\frac{(2j-1)\pi}{2(n-1)} & (j=1,\ldots,n-1)\\ \pi/2\end{cases}$
E_6	$\pi/6$	$\begin{cases}m_1=m_5=\sqrt{\frac{3}{2}} \quad m_2=m_4=\frac{1}{2}(3+\sqrt{3})\\ m_3=\frac{3+\sqrt{3}}{\sqrt{2}} \quad m_6=\sqrt{3}\end{cases}$	$\begin{cases}(\pi/12, \; 5\pi/12,\\ 7\pi/12, \; 11\pi/12,\\ \pi/3, \; 2\pi/3)\end{cases}$
E_7	$\pi/9$	$\begin{cases}m_1=1.347\,29\ldots\\ m_2=2m_1\cos\pi/18 \quad m_3=\frac{1}{2}m_1/\sin(\pi/18)\\ m_4=4m_1\cos\pi/18\cos\pi/9 \; m_5=4m_1\cos\pi/18\sin 2\pi/9\\ m_6=2m_1\sin 2\pi/9 \quad m_7=2m_1\cos\pi/9\end{cases}$	$\begin{cases}(\pi/18,\\ 5\pi/18, \; 7\pi/18,\\ 9\pi/18, \; 11\pi/18,\\ 13\pi/18, \; 17\pi/18)\end{cases}$
E_8	$\pi/15$	$\begin{cases}m_1=2.0361\ldots\\ m_2=2m_1\cos\pi/30 \quad m_3=m_1\sin(\pi/10)/\sin(\pi/30)\\ m_4=m_1\sin(2\pi/15)/\sin(\pi/30) \quad m_5=\frac{1}{2}m_1/\sin(\pi/30)\\ m_6=m_1\cos(\pi/30)/\sin(\pi/10) \quad m_7=\frac{1}{2}m_1/\sin(\pi/10)\\ m_8=\frac{1}{2}m_1/\sin(\pi/15)\end{cases}$	$\begin{cases}(\pi/30,\\ 7\pi/30, \; 11\pi/30,\\ 13\pi/30, \; 17\pi/30,\\ 19\pi/30, \; 23\pi/30,\\ 29\pi/30)\end{cases}$

where the jth component of $\boldsymbol{w}$ is $m_j \exp(-\kappa\Lambda)$ (κ and m_j are presented in table 1) and $\boldsymbol{P}$ stands for

$$\boldsymbol{P}(x) = -\boldsymbol{u}(x) + \mathrm{i}\mathscr{D}\mathscr{G}'\mathscr{D}\boldsymbol{s}/12N^2 \tag{29}$$

where $(\mathscr{G})_{i,j} = \delta_{i,j} g_1^i$, $(s)_j = 1/\sigma_n^{(j)}(\Lambda)$.

The magnitude of $\sigma_N^{(j)}(\Lambda)$ can be fixed by (11), or equivalently by

$$\tilde{X}_+^j(x=0) = \frac{1}{2N} + \frac{1}{2N} \sum_k \mathscr{C}_{jk} \boldsymbol{n}_k \tag{30a}$$

and by

$$X_+^j(\lambda = 0) = \tfrac{1}{2}\sigma_N^{(j)}(\Lambda). \tag{30b}$$

The set of equations (30a) and (30b) leads to a lengthy vector equation which $\sigma_N^{(j)}(\Lambda)$ must satisfy. Combining it with (13), we have

$$f_N(\theta) - f_\infty(\theta) = \pi \sin(\kappa\theta)\left(-\frac{r}{6N^2} + \frac{1}{2N^2}\,{}^t\boldsymbol{n}\mathscr{C}\boldsymbol{n}\right). \tag{31}$$

After imposing the proper normalisation on the free energy [12], we employ the formulae derived in [1]-[3] and obtain a universal expression for the dimensions of scaling operators as

$$X_{n,0} = \tfrac{1}{4}\,{}^t\boldsymbol{n}\mathscr{C}\boldsymbol{n}. \tag{32}$$

The central charge is also

$$c = r \tag{33}$$

which shows that the degree of freedom is equal to the rank of the Lie algebra; in other words all excitations are massless [12]. If there exist 'hole' excitations, the above expression must be slightly modified to

$$X_{n,m} = \tfrac{1}{4}\,{}^t\boldsymbol{n}\mathscr{C}\boldsymbol{n} + {}^t\boldsymbol{m}\mathscr{C}^{-1}\boldsymbol{m}. \tag{34}$$

Similarly we also have the spin dimensions as

$$S_{n,m} = \boldsymbol{n} \cdot \boldsymbol{m}. \tag{35}$$

We may assert that the $\boldsymbol{n}$ correspond to the 'spin wave' excitation and the $\boldsymbol{m}$ correspond to the 'vortex' excitation. Thus we can say that expressions (33) and (34) are natural extensions to those of SU(2) (or the Coulomb gas) [17]. Since the 2D Coloumb gas is equivalent to the critical SU(2) model, one may argue that the counterpart to the NBA model is, say, a multicomponent 2D Coulomb gas.

The details of the present letter will be published elsewhere.

I am grateful to Professor T Izuyama, Dr H Namaizawa and Mr A Kuniba for comments and advice.

Note added. After submission of this letter, I received a preprint from Professor H J de Vega [18] in which he obtained similar results in the case of the $q(2q-1)$-vertex model, which possesses the trigonometric S matrix and A_{q-1} symmetry. I thank Professor de Vega for sending that preprint prior to publication.

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